

ISOPERIMETRIC INEQUALITIES AND RATIONAL HOMOTOPY INVARIANTS

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ABSTRACT. We estimate the linear isoperimetric constants of an n -dimensional ellipse. Using these estimates and a technique of Gromov, we estimate the Hopf and linking invariants of Lipschitz maps from ellipses to round spheres. Using these estimates, we give a lower bound for the k -dilation of degree non-zero maps between ellipses.

We will discuss the following question relating geometry and topology.

Question. *Suppose that F is a map from (S^3, g) to (S^2, h) . If F has Lipschitz constant L , how big can the Hopf invariant of F be? In particular, what is the asymptotic behavior as $L \rightarrow \infty$?*

In this paper, we survey what is known about this question. We begin with some general results due to Gromov and then work out some examples, especially the example of 3-dimensional ellipses. After that, we consider some generalizations of this question. The paper contains a mix of exposition and new work.

Gromov looked at this question in [6] and [8] and proved two basic results. The first result deals with the case that g and h are the standard round metrics.

Proposition 1. *(Gromov, [8], page 358) If F is a map from (S^3, stan) to (S^2, stan) with Lipschitz constant L , then the Hopf invariant of F has norm at most CL^4 . On the other hand, for any $L > 2$, there exist maps with Lipschitz constant L and Hopf invariant at least cL^4 .*

Gromov generalized this result to deal with other metrics (S^3, g) , giving an estimate involving isoperimetric information about the metric g . To state Gromov's estimate, we need a little vocabulary. If z is a closed integral 1-cycle in (S^3, g) , the filling volume of z is the smallest volume of any integral 2-chain y with $\partial y = z$. The isoperimetric constant $Iso(g)$ is the supremal ratio $FillVol(z)/Vol(z)$ as z varies over all closed 1-cycles in (S^3, g) .

Proposition 2. *(Gromov, [6], page 96) Let $F : (S^3, g) \rightarrow (S^2, h)$ be a map with Lipschitz constant L . Then the Hopf invariant of F obeys the following inequality.*

$$|Hopf(F)| \leq Iso(g)Vol(g)Area(h)^{-2}L^4. \quad (*)$$

Given a metric (S^3, g) , it is not trivial to estimate $Iso(g)$. The main new result in this paper is an estimate for $Iso(g)$ when g is the metric of a 3-dimensional ellipse.

We recall that an n -dimensional ellipse with principal axes $E_0 \leq \dots \leq E_n$ is the set $\{x \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n (x_i/E_i)^2 = 1\}$. An n -dimensional ellipse is one of the simplest examples of a Riemannian metric on S^n .

Proposition 3. *Suppose that E is a 3-dimensional ellipse with principal axes $E_0 \leq \dots \leq E_3$. Then, up to a factor of 100, $Iso(E) \sim E_2$.*

If g is the metric of an ellipse E and h is the standard metric, then the upper bound (*) tells us that the Hopf invariant of F has norm at most $CL^4 Vol(E) E_2 \sim CL^4 E_1 E_2^2 E_3$. This upper bound is essentially sharp.

Proposition 4. *Suppose that g is an ellipsoidal metric on S^3 with principal axes $E_0 \leq \dots \leq E_3$. If $L > CE_1^{-1}$, then there is a map F from (S^3, g) to $(S^2, stan)$ with Hopf invariant at least $cL^4 E_1 E_2^2 E_3$.*

These matching upper and lower bounds give us an accurate idea of how the dimensions of the ellipse E relate to the largest Hopf invariant of a Lipschitz map from E .

In general the upper bound (*) may be far from sharp. We contrive an example where (*) is far from sharp by gluing together two differently shaped ellipses.

Next we turn to generalizations of this question in higher dimensions. The Hopf invariant is defined for maps from S^{4n-1} to S^{2n} for each integer $n \geq 1$. The propositions above have analogues in every dimension. But I have decided instead to focus on the linking invariant. The linking invariant is a homotopy invariant for maps from S^n to a wedge of spheres $S^{k_1} \vee S^{k_2}$. It is well-defined whenever the dimensions satisfy $n+1 = k_1 + k_2$ and each k_i is at least 2. The linking invariant is defined in the following way. Let q_1 be a generic point in S^{k_1} and let q_2 be a generic point in S^{k_2} . Consider the inverse images $F^{-1}(q_1)$ and $F^{-1}(q_2)$ in S^n . These are each oriented submanifolds of S^n , and the condition on the dimension guarantees that the two submanifolds have a well-defined linking number. This number is the linking invariant of F . The linking invariant is closely related to the Hopf invariant, but there are more linking invariants to play with, because n can take any value and for each n we have several choices of k_1 and k_2 .

All of our previous results generalize to linking invariants. (Gromov's results generalize to all rational homotopy invariants, as discussed in Chap. 7 of [8].) For the rest of this introduction, we fix n , k_1 , and k_2 , with the convention that $k_1 + k_2 = n+1$ and $2 \leq k_1 \leq k_2$.

Proposition 1A. *(Gromov) Equip S^n and $S^{k_1} \vee S^{k_2}$ with the standard metrics. Suppose that $F : S^n \rightarrow S^{k_1} \vee S^{k_2}$ has Lipschitz constant L . Then the linking invariant of F has norm at most $C(n)L^{n+1}$. If L is larger than $C(n)$, then there are examples with linking invariant at least $c(n)L^{n+1}$.*

To state the higher-dimensional version of Proposition 2, we need a little more vocabulary. Suppose that z is an exact integral k -cycle in a Riemannian manifold (M^n, g) . The filling volume of z is the smallest volume of any integral $(k+1)$ -chain y with $\partial y = z$. We define the isoperimetric constant $Iso_k(M^n, g)$ to be the supremal ratio $FillVol(z)/Vol(z)$ as z varies over all exact k -cycles in M .

Proposition 2A. *(Gromov) Equip S^n with a metric g , and equip $S^{k_1} \vee S^{k_2}$ with a metric h_1 on S^{k_1} and h_2 on S^{k_2} . Suppose that $F : S^n \rightarrow S^{k_1} \vee S^{k_2}$ has Lipschitz constant L . Then the linking invariant of F is bounded as follows.*

$$|L(F)| \leq Iso_{n-k_1}(S^n, g) Vol(g) Area(h_1)^{-1} Area(h_2)^{-1} L^{n+1}.$$

In order to apply this upper bound, we have to estimate the isoperimetric constant of (S^n, g) . This is not trivial in general. In this paper we carry it out for n -dimensional ellipses.

Proposition 3A. *Let E denote the n -dimensional ellipse with principal axes $E_0 \leq \dots \leq E_n$. Up to a constant factor $C(n)$, the isoperimetric constant $Iso_k(E)$ is given by the following formula.*

$$Iso_k(E) \sim E_{k+1} + E_{n-k}.$$

Proposition 3A is the main new result in this paper. To get a sense for it, let's consider some k -cycles z in E . First we consider the smallest k -dimensional equator of E . This equator is the intersection of E with the $(k+1)$ -plane spanned by the x_0, \dots, x_k coordinates. This equator is a k -dimensional ellipse with principal axes $E_0 \leq \dots \leq E_k$ and volume roughly $E_1 \dots E_k$. Suppose now that we intersect E with the $(k+2)$ -plane spanned by the first $k+2$ coordinates. The intersection is a $(k+1)$ -dimensional ellipse with volume roughly $E_1 \dots E_{k+1}$, and z is a hypersurface in the intersection, dividing it into two equal pieces. Therefore, the filling volume of z is at most $\sim E_1 \dots E_{k+1}$. But it's not too hard to convince oneself that this filling is near-optimal, so that the filling volume of z really is $\sim E_1 \dots E_{k+1}$. In this case $FillVol(z)/Vol(z)$ is roughly E_{k+1} .

Second, we consider the largest k -dimensional equator of E . This equator is the intersection of E with the $(k+1)$ -plane spanned by the x_{n-k}, \dots, x_n coordinates. It is an ellipse with principal axes E_{n-k}, \dots, E_n and volume roughly $E_{n-k+1} \dots E_n$. By the same argument as above, it bounds a $(k+1)$ -dimensional hemisphere with volume $\sim E_{n-k} \dots E_n$, and it's not hard to see that its filling volume is $\sim E_{n-k} \dots E_n$. In this case, the ratio $FillVol(z)/Vol(z)$ is roughly E_{n-k} .

This argument shows that $Iso_k(E)$ is at least $c(n)[E_{k+1} + E_{n-k}]$. The proposition claims that $Iso_k(E)$ is not much bigger than this. In other words, the proposition tells us that an arbitrary cycle z in E is not substantially harder to fill than one of the equators of E .

Combining Proposition 2A and Proposition 3A, we get an upper bound for the linking invariant of a Lipschitz map from E , and the upper bound turns out to be essentially sharp.

Proposition 4A. *Let E be an n -dimensional ellipse with principal axes $E_0 \leq \dots \leq E_n$. Equip $S^{k_1} \vee S^{k_2}$ with the standard metrics. Suppose that F is a map from E to $S^{k_1} \vee S^{k_2}$ with Lipschitz constant L . Then the linking invariant of F has norm at most $C(n)L^{n+1}E_{n-k_1+1}E_1 \dots E_n$. On the other hand, if $L > C(n)E_1^{-1}$, then there exists a map with Lipschitz constant L and linking invariant at least $c(n)L^{n+1}E_{n-k_1+1}E_1 \dots E_n$.*

I became interested in this subject because of my work on k -dilation. The k -dilation is a generalization of the Lipschitz constant which measures how much a mapping stretches k -dimensional volumes. If F is a smooth map between Riemannian manifolds, then $Dil_k(F)$ can be defined as the supremum of $|\Lambda^k dF|$. It can also be described more geometrically as follows. We say that F has k -dilation at most Λ if, for every k -dimensional surface Σ^k in the domain, the volume of $F(\Sigma^k)$ is at most Λ times the volume of Σ^k . If F has Lipschitz constant L , then it follows easily that it also has k -dilation at most L^k for every k . On the other hand, a map with 2-dilation 1 may have arbitrarily large Lipschitz constant. A map with 2-dilation

1 may stretch lengths by an arbitrarily large amount in one direction as long as it compensates by contracting in other directions.

Gromov was one of the first mathematicians to look at k -dilation. In particular, he noticed that the proofs of the propositions above don't need the hypothesis $Lip(F) \leq L$, but only the hypothesis $Dil_k(F) \leq L^k$ for an appropriate value of $k > 1$. For example, Proposition 2A has the following generalization.

Proposition 2B. *(Gromov) Equip S^n with a metric g , and equip $S^{k_1} \vee S^{k_2}$ with a metric h_1 on S^{k_1} and h_2 on S^{k_2} . Suppose that $F : S^n \rightarrow S^{k_1} \vee S^{k_2}$ has k_1 -dilation at most L^{k_1} . Then the linking invariant of F is bounded as follows.*

$$|L(F)| \leq Iso_{n-k_1}(S^n, g) Vol(g) Area(h_1)^{-1} Area(h_2)^{-1} L^{n+1}.$$

Similarly, there is a generalization of Proposition 4A using k_1 -dilation instead of the Lipschitz constant. This estimate in turn allows us to bound the k -dilation of maps from one ellipse to another.

Proposition 5. *Let E be an n -dimensional ellipse with principal axes $E_0 \leq \dots \leq E_n$, and let E' be an n -dimensional ellipse with principal axes $E'_0 \leq \dots \leq E'_n$.*

Suppose that F is a map from E to E' with degree D . Suppose that $k \leq (n+1)/2$. Let Q_i denote the quotient E'_i/E_i .

$$\text{Then } [Dil_k(F)]^{\frac{n+1}{k}} \geq c(n) |D| Q_{n+1-k} \prod_{i=1}^n Q_n.$$

This paper contains some new results, but it's also partly expository. I hope that it will provide some useful background for more difficult papers giving estimates for rational homotopy invariants.

One other class of metrics (S^3, g) for which our initial question has been studied are the Berger spheres. In [7], Gromov gives some estimates for the Hopf invariant of a map from a Berger sphere. More generally, he gives estimates for the rational homotopy invariants of Lipschitz maps from Carnot-Caratheodory spaces. The Berger sphere (S^3, g_t) is given by stretching the fibers of the Hopf fibration to have length $2\pi t$, while keeping the metric unchanged in the directions perpendicular to the fibers. If $t < 1$, then Proposition 2 gives a good estimate for the Hopf invariant, but if t is very large then it gives a bad estimate. In [7], Gromov proves a better upper bound. In [4], we show that Gromov's improved upper bound is sharp up to a constant factor.

The paper [3] studies second-order rational homotopy invariants. The methods in this paper extend immediately to give upper bounds for second-order homotopy invariants, but the resulting upper bounds are much too big even for ellipses. In [3], we give sharp estimates (up to a constant factor) for the second-order linking invariant of a Lipschitz map from an n -dimensional ellipse to a wedge of unit spheres.

The paper [2] studies the Hopf invariant of continuous maps to a surface of genus $g \geq 2$. We prove that if M is an oriented 3-manifold that can be triangulated with N simplices, then the Hopf invariant of any map from M to a surface of genus 2 is bounded by C^N . First, we modify a construction of Milnor-Thurston in order to homotope an arbitrary continuous map to one with a bound on the 2-dilation. Then we use the methods from this paper to bound the Hopf invariant using the 2-dilation.

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1. GROMOV'S PROOF

We begin by proving Gromov's upper bound for the Hopf invariant.

Proposition 1.1. *(Gromov, [6], pages 96-97) Let $F : (S^3, g) \rightarrow (S^2, h)$ be a C^1 map with 2-dilation L^2 . Then the Hopf invariant of F is at most the following expression.*

$$|\text{Hopf}(F)| \leq \text{Iso}_1(g) \text{Vol}(g) \text{Area}(h)^{-2} L^4.$$

Remarks: In particular, if the map F has Lipschitz constant L , then it has 2-dilation at most L^2 and the conclusion follows. Also, if g and h happen to be the standard metrics, we get $|\text{Hopf}(F)| \leq 100L^4$.

Proof. The first step of the proof is to observe that if F has small 2-dilation, then one of the fibers $F^{-1}(y)$ has small length. This follows from the coarea formula. We record this fact as a lemma. For future reference, we phrase the lemma using general dimensions.

Lemma 1.1. *Suppose that $F : (M^d, g) \rightarrow (N^q, h)$ is a C^∞ smooth map with q -dilation Λ , where $d \geq q$. Then F has a fiber with $(d-q)$ -dimensional volume at most $\text{Vol}(g)\Lambda/\text{Vol}(h)$.*

Proof. First we write down the coarea formula.

$$\int_M \text{Jac}[dF(x)] d\text{vol}_g = \int_N \text{Vol}^{d-q}[F^{-1}(y)] d\text{vol}_h(y).$$

The quantity $\text{Jac}[dF(x)]$ is exactly the q -dilation of F at the point x , which is at most Λ . According to Sard's theorem, almost every fiber of F is regular. In particular, they are almost all manifolds of dimension $d - q$. The quantity $\text{Vol}^{d-q}[F^{-1}(y)]$ is the $(d-q)$ -dimensional volume of the manifold $F^{-1}(y)$.

It follows that $\int_N \text{Vol}^{d-q}[F^{-1}(y)] d\text{vol}_h(y) \leq \Lambda \text{Vol}(M, g)$, and so we can choose a regular value y so that the fiber $F^{-1}(y)$ has volume at most $\Lambda \text{Vol}(g)/\text{Vol}(h)$. \square

At this point, we make some remarks about the regularity of F . The coarea formula holds for any C^1 map. However, for a C^1 map, the fibers may not be generically manifolds. For a C^1 map, the coarea formula involves the Hausdorff measure of the fibers, and we can find a fiber with $(d-q)$ -dimensional Hausdorff measure obeying the bound in the lemma.

On the other hand, any C^1 map with q -dilation less than Λ may be smoothed to give a homotopic C^∞ map with q -dilation less than Λ . So it suffices to prove the proposition for C^∞ maps. From now on we assume that F is C^∞ .

Returning to the proof of Gromov's estimate, we see that F has a regular fiber $F^{-1}(y)$ with length at most $\text{Vol}(g)L^2/\text{Area}(h)$. This fiber bounds a 2-chain C with area at most $\text{Vol}(g)L^2\text{Iso}_1(g)/\text{Area}(h)$. The map F restricts to a map from $(C, \partial C)$ to (S^2, y) . This map has a well-defined degree which is equal to the Hopf invariant of F . On the other hand, the degree of this map is clearly bounded by $\text{Area}(C)L^2/\text{Area}(h)$. Plugging in our bound for the area of C , we see that the degree is bounded by $\text{Vol}(g)\text{Iso}_1(g)L^4/\text{Area}(h)^2$. \square

The factor L^4 in Gromov's estimate has the right exponent because of the following example. (I believe that this example is due to Gromov, but it may be older.)

Proposition 1.2. *For large L , there is a map F from the unit 3-sphere to the unit 2-sphere with Lipschitz constant L and Hopf invariant on the order of L^4 .*

Proof. Let $H : S^3 \rightarrow S^2$ denote the Hopf fibration. It has Lipschitz constant 2. Let $F : S^2 \rightarrow S^2$ denote a map of Lipschitz constant $L/2$. For large L , the map F may have degree D on the order of L^2 . The composition $H \circ F$ has Hopf invariant D^2 on the order of L^4 . \square

2. LINKING INVARIANTS

In this section, we recall the definition of the linking invariant. Gromov's argument directly applies to give estimates for the linking invariant.

Suppose that $k_1 \leq k_2$ and $n = k_1 + k_2 - 1$. Let F be a smooth map from S^n to the wedge of spheres $S^{k_1} \vee S^{k_2}$. Then we define the linking invariant $L(F)$ to be the linking number of the fibers $F^{-1}(y_1)$ and $F^{-1}(y_2)$ for regular points $y_1 \in S^{k_1}$ and $y_2 \in S^{k_2}$.

Here is an equivalent definition. Let y_1 be a regular point in S^{k_1} . Let C be a k_2 chain in S^n with boundary $F^{-1}(y_1)$. Let $\pi : S^{k_1} \vee S^{k_2} \rightarrow S^{k_2}$ collapse S^{k_1} to the basepoint and be equal to the identity on S^{k_2} . Then $\pi \circ F$ maps $(C, \partial C)$ to $(S^{k_2}, *)$, where $*$ denotes the basepoint. We define $L(F)$ to be the degree of this map.

Standard arguments show that $L(F)$ is independent of the choice of y_1 and y_2 and that it is a homotopy invariant. (These arguments are completely analogous to the arguments about the Hopf invariant which appear in the excellent reference [1].)

Gromov's argument immediately extends to all linking invariants.

Proposition 2.1. *Suppose $k_1 \leq k_2$ and $k_1 + k_2 = n + 1$. Let $F : (S^n, g) \rightarrow (S^{k_1} \vee S^{k_2}, h_1 \vee h_2)$ be a map with k_1 -dilation L^{k_1} . Then the linking invariant of F is at most the following expression.*

$$|L(F)| \leq Iso_{k_2-1}(g) Vol(g) Area(h_1)^{-1} Area(h_2)^{-1} L^{n+1}.$$

Remark on notation: In the statement of the theorem, h_1 denotes a metric on S^{k_1} and h_2 denotes a metric on S^{k_2} .

Proof. Let $U \subset S^n$ denote $F^{-1}(S^{k_1})$. Clearly the volume of U is at most the volume of (S^n, g) . Applying Lemma 1.1 to the map $F : U \rightarrow S^{k_1}$, we find a fiber $F^{-1}(y_1)$ with $k_2 - 1$ -dimensional volume at most $Vol(g) L^{k_1} / Area(h_1)$. This fiber bounds a k_2 -dimensional chain C in S^n of volume at most $Vol(g) L^{k_1} Iso_{k_2-1}(g) / Area(h_1)$. Now we consider the map $\pi \circ F$ from $(C, \partial C)$ to $(S^{k_2}, *)$. The map π has Lipschitz constant 1. The map F has k_2 -dilation at most L^{k_2} since $k_2 \geq k_1$. (This follows from linear algebra, see Appendix 1 of [5].) Therefore, $\pi \circ F$ has degree at most $Vol(C) L^{k_2} / Area(h_2)$. Plugging in our bound for $Vol(C)$ finishes the proof. \square

3. THE ISOPERIMETRIC CONSTANTS OF ELLIPSES

In this section, we estimate the isoperimetric constants of ellipses. This section contains the main work in the paper.

Proposition 3.1. *If E is an n -dimensional ellipse with principal axes $E_0 \leq \dots \leq E_n$, then $\text{Iso}_k(E) < C(n)[E_{k+1} + E_{n-k}]$.*

We will prove in Section 5 that this upper bound is sharp up to a constant factor $C(n)$. The proof uses the k -dimensional equators of E which we discussed in the introduction.

Proof. The ellipse E is $C(n)$ -bilipschitz equivalent to the double of a rectangle R with dimensions $R_1 \leq \dots \leq R_n$, with $R_i = E_i$. Each copy of R in the double corresponds to a hemisphere of E .

Let z be an integral k -cycle in E . We let z_N denote the part of z in the Northern hemisphere and z_S denote the part of z in the Southern hemisphere. The chain z_S is a relative cycle in the rectangle R . Our first step is to push z_S to the boundary of R . More formally, we find a $(k+1)$ -chain y_S in R with $\partial y_S = z_S + B$ and B in the boundary of R . The chain B lies on the equator of E , so we can also think of it as a chain in the northern hemisphere. Now $z_N - B$ forms a new absolute cycle in the Northern hemisphere. The second step is to fill this absolute cycle with a chain y_N . Finally, we take $y = y_N + y_S$. It follows immediately that $\partial y = z_N + z_S = z$. Our goal is to construct y_N and y_S with the given bounds on the volume.

The good thing about this approach is that it divides our problem into two subproblems, and each subproblem takes place in a rectangle in Euclidean space. Rectangles are easier to work with than ellipses in my experience. This approach, however, leads immediately to a serious problem: the chain B may be much larger than the cycle z .

This point is the main difficulty in the paper. For example, suppose that z is the smallest k -dimensional equator. The intersection of z with each hemisphere is a k -dimensional rectangle with dimensions $R_1 \times \dots \times R_k$ centered in the middle of R . A natural choice for y is given by two copies of a rectangle with dimensions roughly $R_1 \times \dots \times R_{k+1}$. The situation is illustrated in Figure 1.

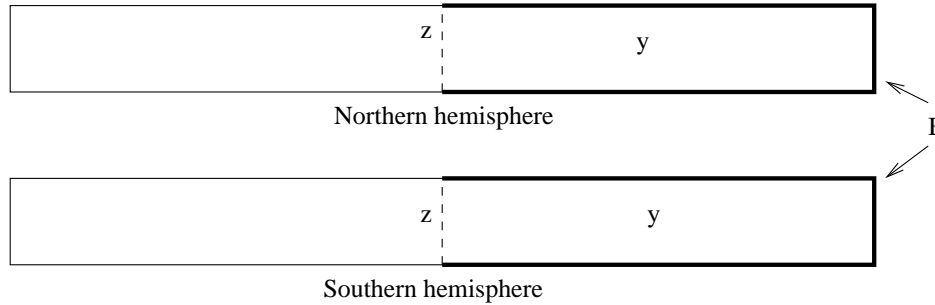


Figure 1. A cycle z and its filling y .

In the picture, the two dotted lines denote the cycle z . The dotted line in the Northern hemisphere denotes z_N and the one in the Southern hemisphere denotes z_S . The chain y is denoted by the labelled rectangles. The rectangle in the Northern hemisphere is y_N , and the rectangle in the Southern hemisphere is y_S . Recall that the dimension of y is $(k+1)$ which may be strictly less than n - so in spite of the picture the reader should not picture y as an open set. The picture shows two copies of the chain B - the chain lies along the equator, and so we can see it in the boundary of each hemisphere. The picture shows that B may be much bigger than z .

In this example, the cycle z has volume $\sim R_1 \dots R_k$, and the chain B has volume $\sim R_2 \dots R_{k+1}$. Consider the last step of our construction: finding y_N . We need to choose y_N with boundary $z_N - B$. One would like to find y_N by applying an isoperimetric inequality for absolute cycles in R . If the volume of B were comparable to the volume of z , then the isoperimetric inequality would deliver a chain y_N with the desired volume bound. But as we have seen the volume of $z_N - B$ is much larger than the volume of z . It is possible to find a cycle w in R with the same volume as $z_N - B$ and with filling volume much larger than our desired volume bound. In order to prove our proposition, we need to take account of some extra structure in the cycle $z_N - B$, which allows us to fill it efficiently. The extra structure comes from keeping track of the "direction" in which volume is pointing.

(It may perhaps be concluded that trying to build y as a sum of a piece in the Northern hemisphere and a piece in the Southern hemisphere is not the right approach to the isoperimetric problem. It would be interesting to see another proof. On the other hand, the directional estimates which we introduce here seem interesting to me. In the paper [3], they are taken further, leading to improved results about second-order rational homotopy invariants.)

We begin by defining directional volume.

Let C be an m -chain in \mathbb{R}^n . Suppose that J is an m -tuple of distinct integers between 1 and n . Let $P(J)$ denote the m -plane with coordinates x_i , $i \in J$. We define the J -volume of C to be the volume (with geometric multiplicity) of the projection of C to $P(J)$. For example, if C is the unit sphere in \mathbb{R}^n and J is any $(n-1)$ -tuple, then the J -volume of C is twice the volume of the unit $(n-1)$ -ball. The total volume of C is roughly equal to the sum of the volumes in different directions.

$$Vol(C) \leq \sum_J Vol_J(C) \leq \binom{n}{m} Vol(C).$$

For an m -tuple J , we define $e(J)$ to be one less than the smallest member of J . For example, if J is $\{3, 6, 8\}$, then $e(J) = 2$. Since J is an m -tuple of the numbers from 1 to n , $0 \leq e(J) \leq n - m$.

In the Northern hemisphere, we will use the following lemma to find y_N .

Lemma 1. *Let z denote an absolute k -cycle in the rectangle R . Then z is the boundary of a chain y obeying the following estimate.*

$$|y| \lesssim \sum_{J|e(J) \geq 1} R_{e(J)} Vol_J(z).$$

Comments. If J contains 1, then $e(J) = 0$, and so $Vol_J(z)$ does not appear on the right-hand side of our inequality. In other words, the volume of y can be bounded using only some of the directional volumes of z .

Proof. Let $\pi(z)$ denote the projection of z to the plane $x_1 = 0$. The cylinder from z to $\pi(z)$ has volume at most $R_1 \sum_{e(J) \geq 1} Vol_J(z)$. The image $\pi(z)$ has the same J -volume as z if J does not contain 1, and it has J -volume zero if J contains 1.

Now we repeat this approach, projecting to the cod-2 plane $x_1 = x_2 = 0$, and so on, until we project to the k -plane $x_1 = \dots = x_{n-k} = 0$. A k -dimensional cycle inside of this k -plane bounds a $(k+1)$ -chain of volume zero, so after making these $(n - k)$ projections we are done.

The projection to the plane of codimension c , $x_1 = \dots = x_c = 0$, costs volume at most $\sum_{e(J) \geq c} R_c \text{Vol}_J(z)$. Summing the contributions we get the inequality we want to prove. \square

In the Southern hemisphere, we need a slightly more complicated lemma.

Lemma 2. *Let z denote a k -cycle in the rectangle R with ∂z contained in ∂R . Then there is a $(k+1)$ -chain y in R so that $\partial y = z + B$, where B is contained in ∂R , obeying the following inequalities.*

$$|y| \lesssim R_{k+1}|z|.$$

$$\text{If } e(J) \geq 1, \text{Vol}_J(y) = 0.$$

$$\text{If } e(J) > 1, \text{Vol}_J(B) \lesssim |z|.$$

$$\text{If } e(J) = 1, \text{Vol}_J(B) \lesssim (R_{k+1}/R_1)|z|.$$

Proof. The proof is by induction on the dimension k of the cycle. When $k = 0$, the lemma is trivial.

We consider the intersection of z with planes $x_1 = h$. We can choose h so that this intersection is a $(k-1)$ -cycle with volume at most $|z|/R_1$. We pick such a value of h and call this intersection z_h .

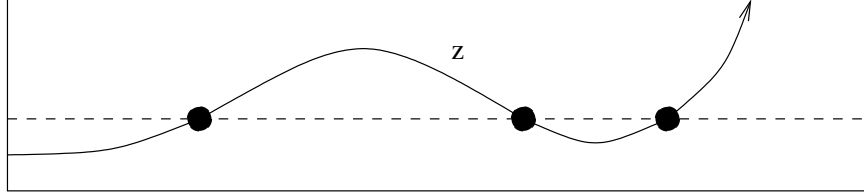


Figure 2. Intersecting a cycle with a plane.

The solid curve in the figure denotes the relative cycle z . The dotted line denotes the plane $x_1 = h$. The three dark points denote their intersection z_h .

We now decompose z into two pieces as follows.

$$z = (z - [0, R_1] \times z_h) + [0, R_1] \times z_h = z_1 + z_2.$$

We deal with the first piece first. We decompose this first piece as $z_+ + z_-$, where z_+ is the part of z_1 lying above the plane $x_1 = h$ and z_- is the part lying below it. The two pieces z_+ and z_- are each relative cycles. We illustrate z_+ and z_- in the next figure.

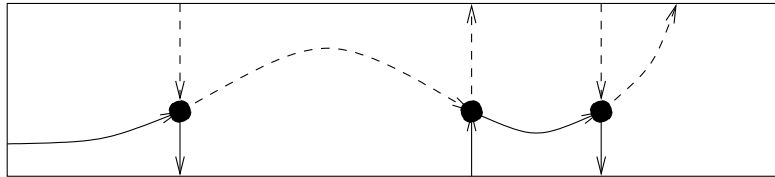


Figure 3. Dividing the cycle z_1 into a top piece and a bottom piece

The dotted curves denote the relative cycle z_+ and the solid curves denote the relative cycle z_- .

Now we construct a filling y_- for z_- . In other words, $\partial y_- = z_- + B_-$ with $B_- \subset \partial R$. We construct y_- by pushing z_- down into the boundary of R . In more detail, we construct a map $F : z_- \times [0, 1] \rightarrow R$. The last $n-1$ coordinates of F are just the coordinates x_i on z_+ . The first coordinate F_1 is given by the formula $F_1(p, t) = tx_1(p)$. Here $x_1(p)$ denotes the coordinate function of the point $p \in z_+$. We define y_- to be the image chain $F([0, 1] \times z_-)$.

We illustrate y_- and B_- below.

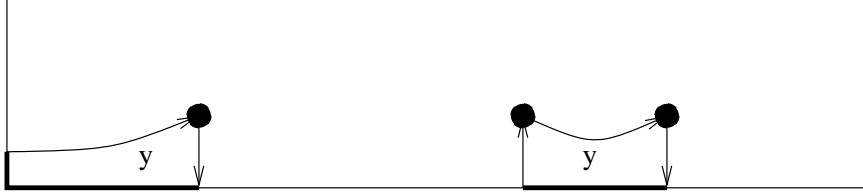


Figure 4. Filling the bottom piece.

The chain y_- is labelled y in the figure. The chain B is the darkened portion of the boundary of the rectangle. The solid oriented curves are again z_- .

The chain y_- has volume at most $R_1|z_-| \leq 2R_1|z|$. Also, if J is a $(k+1)$ -tuple that does not contain 1, then $Vol_J(y_-) = 0$. Therefore, y_- obeys the conclusions of the lemma.

We define B_- to be $\partial y_- - z_-$. Equivalently, B_- is $-F(\{0\} \times z_-) + F([0, 1] \times \partial z_-)$. Since ∂z_- lies in the bottom and sides of R , not touching the top face of R , B_- is contained in ∂R . The first term in B_- , $F(\{0\} \times z_-)$, is just the projection of z_- to the plane $x_1 = 0$. It has volume at most $|z|$. The second part may have arbitrarily large volume, but if J does not contain 1, then its J -volume is zero. Therefore B_- obeys the conclusions of the lemma.

Similarly, we define y_+ and B^+ . Combining them, we get a filling for z_1 . Now we turn to $z_2 = [0, R_1] \times z_h$. The relative cycle z_h has dimension $k-1$ and lives inside an $(n-1)$ -dimensional rectangle with dimensions $R_2 \times \dots \times R_n$. By induction, we can find a filling y_h for z_h obeying the conclusion of the lemma.

We define $y_2 = [0, R_1] \times y_h$. The volume of y_2 is at most $R_1|y_h|$. By our inductive hypothesis, $|y_h| \leq R_{k+1}|z_h|$. Finally, the volume of y_2 is at most $R_1R_{k+1}|z_h| \leq R_{k+1}|z|$. Also, if J does not contain 1, then the J -volume of y_2 is zero. So y_2 obeys the conclusion of the lemma.

We define $B_2 = \partial y_2 - z_2 = \{R_1\} \times y_h - \{0\} \times y_h + [0, R_1] \times B_h$. By induction, we know that the J -volume of y_h vanishes unless J contains 2. Also, if J does not contain 1, then the J -volume of $[0, R_1] \times B_h$ is zero. Therefore, if $e(J) > 1$, then the J -volume of B_2 is zero. If $e(J) = 1$, then the J -volume of B_2 is at most $2|y_h| \lesssim R_{k+1}|z_h| \leq (R_{k+1}/R_1)|z|$. Therefore, B_2 obeys the conclusion of the lemma.

Our filling of z is $y = y_1 + y_2$. We have $\partial y = z + B$ where $B = B_1 + B_2$. Because of our estimates for y_i and B_i , y and B obey the conclusions of the lemma. \square

Combining the two lemmas, we can finish the proof of the proposition. Let z be a k -cycle in the ellipse E . Let z_S be the intersection of z with the Southern hemisphere. By Lemma 2, we can find a chain y_S in the Southern hemisphere so that $\partial y_S = z_S + B$, obeying the following estimates. First, $|y_S| \lesssim E_{k+1}|z|$. Second, if $e(J) = 1$, then $Vol_J(B) \lesssim (E_{k+1}/E_1)|z|$. Third, if $e(J) > 1$, then $Vol_J(B) \lesssim |z|$. Since B is in the equator, we can view it as belonging to the Northern hemisphere.

Let z_N be the intersection of z with the Northern hemisphere. Then $z_N - B$ is an absolute k -cycle in the Northern hemisphere. We use Lemma 1 to find a chain y_N in the Northern hemisphere with $\partial y_N = z_N - B$. According to Lemma 1, the volume of y_N is bounded by the following expression.

$$\sum_{J|e(J) \geq 1} E_{e(J)} \text{Vol}_J(z_N - B).$$

If $e(J) > 1$, then $\text{Vol}_J(z_N - B) \lesssim |z|$, so the contribution of these terms is $\lesssim E_{n-k}|z|$. If $e(J) = 1$, then $\text{Vol}_J(z_N - B) \lesssim (E_{k+1}/E_1)|z|$, and so the contribution of those terms is $\lesssim E_{k+1}|z|$.

Finally, we define $y = y_S + y_N$, a $(k+1)$ -cycle with $\partial y = z$. The volume of y is $\lesssim (E_{k+1} + E_{n-k})|z|$. Hence $\text{Iso}_k(E) \leq C(n)[E_{k+1} + E_{n-k}]$. \square

4. UPPER BOUNDS FOR HOMOTOPY INVARIANTS OF LIPSCHITZ MAPS

In Section 3, we estimated the isoperimetric constants of ellipses. If we plug these estimates into Propositions 1.1 and 2.1, we get upper bounds for the Hopf and linking invariants of Lipschitz maps from ellipses.

The estimate for the Hopf invariant is as follows.

Proposition 4.1. *If E is a 3-dimensional ellipse and $F : E \rightarrow (S^2, h)$ is a map with 2-dilation L^2 then the Hopf invariant of F is at most $C(n)E_2 \text{Vol}(E)L^4/\text{Area}(h)^2$. In particular, if h is the unit sphere metric, then the Hopf invariant of F is at most $C(n)E_1E_2^2E_3L^4$.*

Proof. By Proposition 1.1,

$$|\text{Hopf}(F)| \leq C \text{Iso}_1(E) \text{Vol}(E) L^4 \text{Area}(h)^{-2}.$$

The volume $\text{Vol}(E)$ is at most $CE_1E_2E_3$. By Proposition 3.1, $\text{Iso}_1(E) < CE_2$. Plugging in we get,

$$|\text{Hopf}(F)| \leq CL^4 \text{Area}(h)^{-2} E_1 E_2^2 E_3.$$

\square

The estimate for the linking invariant is as follows.

Proposition 4.2. *Suppose that E is an n -dimensional ellipse with principal axes $E_0 \leq \dots \leq E_n$. Suppose that $n+1 = k_1 + k_2$ and $2 \leq k_1 \leq k_2$. If $F : E \rightarrow (S^{k_1} \vee S^{k_2}, h_1 \vee h_2)$ is a map with k_1 -dilation L^{k_1} , then the linking invariant of F is at most $C(n)E_{k_2} \text{Vol}(E)L^{n+1} \text{Area}(h_1)^{-1} \text{Area}(h_2)^{-1}$. In particular, if each h_i is the unit sphere metric, then the linking invariant of F is at most $C(n)E_{k_2} \text{Vol}(E)L^{n+1}$.*

Proof. By Proposition 2.1,

$$|L(F)| \leq \text{Iso}_{k_2-1}(E) \text{Vol}(E) \text{Area}(h_1)^{-1} \text{Area}(h_2)^{-1} L^{n+1}.$$

The volume of E is at most $CE_1 \dots E_n$. By Proposition 3.1, the isoperimetric constant $\text{Iso}_{k_2-1}(E)$ is at most $C[E_{k_2} + E_{n-k_2+1}]$. This expression is equal to $C[E_{k_2} + E_{k_1}]$. By assumption, $k_1 \leq k_2$, so we conclude the following.

$$|L(F)| \leq CL^{n+1} \text{Area}(h_1)^{-1} \text{Area}(h_2)^{-1} E_{k_2} E_1 \dots E_n.$$

\square

5. LIPSCHITZ MAPS WITH LARGE HOPF OR LINKING INVARIANTS

The upper bounds in Proposition 4.1 and 4.2 are sharp up to a constant factor. To prove this, we construct Lipschitz mappings with large homotopy invariants.

Proposition 5.1. *If E is a 3-dimensional ellipse and $L > CE_1^{-1}$, then there exists a map F from E to the unit 2-sphere with Lipschitz constant L and with Hopf invariant at least $cE_2\text{Vol}(E)$.*

Proof. We recall that E is C -bilipschitz to the double of the rectangle R with dimensions $E_1 \times E_2 \times E_3$. Inside of the double, we consider a set U equal to the double of $[0, E_1] \times [E_2/3, 2E_2/3] \times [E_3/3, 2E_3/3]$. Let π denote the projection from U to $U' = [E_2/3, 2E_2/3] \times [E_3/3, 2E_3/3]$. Let f_1 denote a map from $(U', \partial U')$ to $(S^2, *)$ with Lipschitz constant L and degree at least $cE_2E_3L^2$.

Let V denote the subset of the Northern hemisphere given by $[0, E_1] \times \tilde{V}$, where \tilde{V} is the region $[0, E_2] \times [0, E_3] - [E_2/3, 2E_2/3] \times [E_3/3, 2E_3/3]$. The region V is C -bilipschitz to a cylinder of the form $S^1(E_3) \times [0, E_1] \times [0, E_2]$. Here $S^1(E_3)$ denotes a circle of radius E_3 . Let π denote the projection from V to $V' = [0, E_1] \times [0, E_2]$. Let f_2 denote a map from $(V', \partial V')$ to $(S^2, *)$ with Lipschitz constant L and degree at least $cE_1E_2L^2$.

Finally, we construct a map $F : E \rightarrow S^2$ by combining f_1 and f_2 . On the set U , we define F to be equal to the map $f_1 \circ \pi$. On the set V , we define F to be equal to $f_2 \circ \pi$. On the remainder of E , we define F to be the basepoint of S^2 .

We claim that the map F has Hopf invariant at least $cE_1E_2^2E_3L^4 \sim cE_2\text{Vol}(E)L^4$.

Topologically, U, V are each thick tubes in S^3 , and the two tubes are linked with linking number 1. For a generic point y in S^2 , the fiber $F^{-1}(y)$ consists of several circles parallel to the core of U together with several circles parallel to the core of V . More precisely, there are at least $cE_2E_3L^2$ circles parallel to the core of U and at least $cE_1E_2L^2$ circles parallel to the core of V . Now consider a second generic point y' in S^2 . The fiber $F^{-1}(y')$ also consists of $\sim E_2E_3L^2$ parallel copies of the core of U and $\sim E_1E_2L^2$ parallel copies of the core of V . The linking number of two parallel copies of the core of U is zero. The same holds for V . The linking number of the core of U with the core of V is equal to 1. Therefore, the linking number of $F^{-1}(y)$ with $F^{-1}(y')$ is at least $cE_1E_2^2E_3L^4$. \square

Proposition 5.2. *If E is an n -dimensional ellipse and $L > C(n)E_1^{-1}$, then there exists a map F from E to the wedge of unit spheres $S^{k_1} \vee S^{k_2}$ with Lipschitz constant L and with linking invariant at least $c(n)E_{k_2}\text{Vol}(E)$.*

Proof. This proof is essentially the same as the last one, which boils down to finding two thick linked tubes.

Again we note that E is $C(n)$ -bilipschitz to the double of a rectangle R with dimensions $E_1 \times \dots \times E_n$.

We define U to be a thick neighborhood of the double of $[0, E_1] \times \dots \times [0, E_{k_2-1}]$. Topologically, U is a neighborhood of a standard copy of S^{k_2-1} embedded in S^n . Geometrically, U is $C(n)$ -bilipschitz to a product $U_1 \times [0, E_{k_2}] \times \dots \times [0, E_n]$, where U_1 is the double of $[0, E_1] \times \dots \times [0, E_{k_2-1}]$. We let π denote the projection from U onto $U' = [0, E_{k_2}] \times \dots \times [0, E_n]$. The dimension of U' is $n - k_2 + 1 = k_1$. Let f_1 denote a map from $(U', \partial U')$ to $(S^{k_1}, *)$ with Lipschitz constant L and degree at least $c(n)E_{k_2} \dots E_n L^{k_1}$.

We define V to be a thick neighborhood of the $(k_1 - 1)$ -dimensional ellipse V_1 sitting in the upper hemisphere, where V_1 is given by the equations $x_1 = \dots = x_{k_2-1} = 0, \sum_{i=k_2}^n [x_i - (1/2)E_i]^2 E_i^{-2} = 1$. The region V is $C(n)$ -bilipschitz to $V_1 \times V'$, for $V' = [0, E_1] \times \dots \times [0, E_{k_2}]$. We let π denote the projection $V \rightarrow V'$. We let f_2 be a map from $(V', \partial V')$ to $(S^{k_2}, *)$ with Lipschitz constant L and degree at least $c(n)E_1 \dots E_{k_2} L^{k_2}$.

Now we define $F : E \rightarrow S^{k_1} \vee S^{k_2}$. We define F on U to be $f_1 \circ \pi$, which maps U to $S^{k_1} \subset S^{k_1} \vee S^{k_2}$. Similarly, we define F on V to be $f_2 \circ \pi$, which maps V to $S^{k_2} \subset S^{k_1} \vee S^{k_2}$. On the rest of E , we define F to be the basepoint of $S^{k_1} \vee S^{k_2}$.

The fiber $F^{-1}(y_1)$ for a generic point $y_1 \in S^{k_1}$ is at least $c(n)E_{k_2} \dots E_n L^{k_1}$ parallel copies of the core of U . The fiber $F^{-1}(y_2)$ for a generic point $y_2 \in S^{k_2}$ is at least $c(n)E_1 \dots E_{k_2} L^{k_2}$ parallel copies of the core of V . The core of U and the core of V have linking number 1. Therefore, the linking invariant of F is at least $c(n)E_{k_2} \text{Vol}(E) L^{n+1}$. \square

This is also a convenient place to show that our upper bounds for $\text{Iso}_k(E)$ are sharp up to a constant factor.

Proposition 5.3. *The isoperimetric constant $\text{Iso}_k(E)$ is at least $c(n)[E_{k+1} + E_{n-k}]$.*

Proof. We let $S \subset E$ denote the smallest k -dimensional equator. We let S' denote the largest $(n-k-1)$ -dimensional equator. The two spheres S and S' are linked with linking number 1. We define V to be a thick neighborhood of S' . The set V is bilipschitz to $S' \times V'$ for $V' = [0, E_1] \times \dots \times [0, E_{k+1}]$. If C is any chain with $\partial C = S$, then the intersection $V \cap C$ has projection that covers V' , and so the volume of C is at least $c(n)E_1 \dots E_{k+1}$. Since the volume of S is roughly $E_1 \dots E_k$, we conclude that $\text{Iso}_k(E)$ is at least $c(n)E_{k+1}$.

Now we let $S \subset E$ denote the smallest $(n-k-1)$ -dimensional equator. We let S' denote the largest k -dimensional equator. The two spheres S and S' have linking number 1. We define U to be a thick neighborhood of S . The set U is bilipschitz to $S \times [0, E_{n-k}] \times \dots \times [0, E_n]$. Therefore, the filling volume of S' is at least $c(n)E_{n-k} \dots E_n$. Since the volume of S' is roughly $E_{n-k+1} \dots E_n$, we conclude that $\text{Iso}_k(E)$ is at least $c(n)E_{n-k}$. \square

6. A BAD EXAMPLE

In this section, we consider a metric more complicated than an ellipsoidal metric. For this example, the upper bound for the Hopf invariant in Proposition 1.1 is much bigger than the optimal upper bound.

Our metric (S^3, g) is a connected sum of the unit 3-sphere with a long thin tube of width $w \ll 1$ and length $A \gg 1$. (The long thin tube looks like $S^2(w) \times [0, A]$, connected on one end to the unit sphere and capped on the other end with a hemispherical cap of radius w .) The volume of the tube is roughly $w^2 A$, and we choose A and w so that $w^2 A$ is much bigger than 1. Regardless of w, A , the isoperimetric constant $\text{Iso}_1(S^3, g)$ is at least ~ 1 . We can consider, for example, a circumference of the unit 3-sphere.

We consider the Hopf invariants of Lipschitz maps from (S^3, g) to the unit 2-sphere. The right-hand side in Proposition 1.1 is $C \text{Iso}_1(g) \text{Vol}(g) \text{Area}(h)^{-2} L^4$, which is at least $\sim A w^2 L^4$. But we will prove that every map with Lipschitz constant L has Hopf invariant much smaller than $A w^2 L^4$.

Proposition 6.1. *Every map from (S^3, g) to the unit 2-sphere with Lipschitz constant L has Hopf invariant at most $C[1 + Aw^3]L^4$.*

Proof. Suppose that F is a map from (S^3, g) to the unit 2-sphere with 2-dilation at most L^2 . Define the thick part of (S^3, g) to be the unit 3-sphere together with the first unit length of the connected tube. The thin part of (S^3, g) is the rest of it. Then we can choose a fiber $F^{-1}(y)$ so that the part of the fiber in the thick part of (S^3, g) has length at most L^2 and the part of the fiber in the thin part has length at most L^2Aw^2 .

Inside the thick part, we included the first unit of the tube, which is bilipschitz to $S^2(w) \times [0, 1]$. We can choose a height $h \in [0, 1]$ so that the fiber $F^{-1}(y)$ meets $S^2(w) \times \{h\}$ in at most L^2 points. Then we can perform surgery at the cut, adding a length at most $L^2w \leq L^2$. After the surgery, we have written the fiber as a sum of 1-cycles $z_1 + z_2$, where z_1 lies in the thick part of (S^3, g) and has length at most $\sim L^2$ and z_2 lies in the thin part of (S^3, g) and has length at most $\sim Aw^2L^2$.

Now we can fill z_1 by a 2-chain C_1 of area at most $\sim L^2$. We can fill z_2 more efficiently because it lies in the thin part of (S^3, g) . Using the argument from Section 3, we can fill z_2 by a 2-chain C_2 of area at most $\sim Aw^3L^2$. Thus we have filled $F^{-1}(y)$ by a chain $C = C_1 + C_2$ of area at most $\sim [1 + Aw^3]L^2$. Hence the Hopf invariant of F is at most $\sim [1 + Aw^3]L^4$. \square

One can choose A, w so that this estimate improves by an arbitrary factor over the upper bound Aw^2L^4 .

7. APPLICATION TO K-DILATION OF MAPS BETWEEN ELLIPSES

Our previous results imply a new lower bound for the k-dilation of a map from one ellipse to another.

Proposition 7.1. *Let E be an n -dimensional ellipse with principal axes $E_0 \leq \dots \leq E_n$, and let E' be an n -dimensional ellipse with principal axes $E'_0 \leq \dots \leq E'_n$. Suppose that F is a map from E to E' with degree D and k -dilation Λ . Suppose that $k \leq (n+1)/2$. Let Q_i denote the quotient E'_i/E_i .*

$$\Lambda^{\frac{n+1}{k}} \geq c(n)|D|Q_{n+1-k} \prod_{i=1}^n Q_n.$$

Proof. We use a map from E' with a large linking invariant. According to Proposition 5.2, there is a map Φ from E' to $S^k \vee S^{n+1-k}$ with Lipschitz constant L and linking invariant at least $c(n)E'_{n+1-k} \text{Vol}(E')L^{n+1}$. Now we consider the composition $\Phi \circ F$ mapping E to $S^k \vee S^{n+1-k}$. This composition has k -dilation at most ΛL^k . Its linking invariant has norm at least $c(n)|D|E'_{n+1-k} \text{Vol}(E')L^{n+1}$. According to Proposition 4.2, the norm of the linking invariant must be at most $C(n)E_{n+1-k} \text{Vol}(E)L^{n+1}\Lambda^{\frac{n+1}{k}}$. \square

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